Classification of large partial plane spreads in $\mathrm{PG}(6,2)$ and related combinatorial objects

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In this article, the partial plane spreads in $\mathrm{PG}(6,2)$ of maximum possible size 17 and of size 16 are classified. Based on this result, we obtain the classification of the following closely related combinatorial objects: Vector space partitions of $\mathrm{PG}(6,2)$ of type $(3^{16}4^1)$, binary 3×4 MRD codes of minimum rank distance 3, and subspace codes with parameters $(7,17,6)_2$ and $(7,34,5)_2$.

MSC Classification

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1 Introduction

In the recent paper [20] constructions and bounds for mixed dimension subspace codes have been studied. It has been announced that the number of isomorphism types of $(7,17,6)_2$ subspace codes is 928 and the number of $(7,34,5)_2$ subspace codes is 20. The details of these classifications are given in this article.

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The main part is the classification of partial plane spreads in $\mathrm{PG}(6,2)$ of maximum size 17 and of size 16. While the full classification is inevitably based on computational methods, we point out that for both sizes the feasible hole structures are classified purely by theoretical arguments. From these results it is comparatively cheap to derive the classification of the above mentioned subspace codes. Moreover, the classification of further, related combinatorial objects is carried out, namely of the vector space partitions of $\mathrm{PG}(6,2)$ of type $(3^{16}4^1)$ and of the (not necessarily linear) binary 3×4 MRD codes of minimum rank distance 3.

Our work can be seen as a continuation of [14], where partial line spreads in $\mathrm{PG}(4,2)$ have been classified. Due to the considerable number of the constructed codes, only the most important ones can be given in explicit form in this article. The full data is provided at the online tables of subspace codes `http://subspacecodes.uni-bayreuth.de`.

The remaining part of the paper is structured as follows. In Section 2 the necessary theoretical background is provided. In particular, partial spreads and the related combinatorial notions of vector space partitions, subspace codes and MRD codes are introduced, together with pointers to the literature. The classification of partial plane spreads in $\mathrm{PG}(6,2)$ of maximum size 17 is carried out in Section 3. The classification of partial plane spreads of the second largest size 16 in Section 4 is considerably more involved, but still feasible. Section 5 contains the implications of these classifications on MRD codes, and Section 6 those on optimal subspace codes.

2 Preliminaries

2.1 The subspace lattice

Throughout this article, V is a vector space over $\mathbb{F}_q$ of finite dimension v. Subspaces of dimension k will be called *k-subspaces* or *$(k-1)$-flats*. The set of all k-subspaces of V is called the *Graßmannian* and denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. The 1-subspaces of V are called *points*, the 2-subspaces *lines*, the 3-subspaces *planes*, the 4-subspaces *solids* and the $(v-1)$-subspaces *hyperplanes*. As usual, subspaces of V are identified with the set of points they contain. The number of all k-subspaces of V is given by the Gaussian binomial coefficient

$$\#\begin{bmatrix} V \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } k \in \{0,\ldots,v\}; \\ 0 & \text{else.} \end{cases}$$

The set $\mathcal{L}(V)$ of all subspaces of V forms the *subspace lattice* of V.

After a choice of a basis, we can identify V with $\mathbb{F}_q^v$. Now for any $U \in \mathcal{L}(V)$ there is a unique matrix A in reduced row echelon such that $U = \langle A \rangle$, where $\langle . \rangle$ denotes the row space. Our focus lies on the case $q = 2$, where the 1-subspaces $\langle \mathbf{x} \rangle_{\mathbb{F}_2} \in \begin{bmatrix} V \\ 1 \end{bmatrix}_2$ are in one-to-one correspondence with the nonzero vectors $\mathbf{x} \in V \setminus \{\mathbf{0}\}$.

By the fundamental theorem of projective geometry, for $v \neq 2$ the automorphism group of $\mathcal{L}(V)$ is given by the natural action of $\mathrm{P\Gamma L}(V)$ on $\mathcal{L}(V)$. The automorphisms contained in $\mathrm{PGL}(V)$ will be called *linear*. The action of these groups provides a notion of (linear) automorphisms and (linear) equivalence on subsets of $\mathcal{L}(V)$ and in particular on partial spreads, subspace codes and vector space partitions, which will be introduced below.

In the case that q is prime, the group $\mathrm{P\Gamma L}(V)$ reduces to $\mathrm{PGL}(V)$, and for the case of our interest $q = 2$, it reduces further to $\mathrm{GL}(V)$. After a choice of a basis of V, its elements are represented by the invertible $v \times v$ matrices A, and the action on $\mathcal{L}(V)$ is given by the vector-matrix-multiplication $\mathbf{v} \mapsto \mathbf{v}A$.

2.2 Partial spreads

A partition of $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ into k-subspaces is called a *k-spread*. Spreads exist if and only $k \mid v$ [1]. Weakening the above definition, a set $\mathcal{S}$ of $(k-1)$-dimensional subspaces of V is called a *partial* $(k-1)$*-spread*, if the intersection of any two elements of $\mathcal{S}$ is trivial. The elements of $\mathcal{S}$ are called *blocks*. The main problem is to determine the maximum possible size of a partial $(k-1)$-spread. A partial plane spread of maximum possible cardinality is called *maximum*. $\mathcal{S}$ is called *extendible* if there is a partial $(k-1)$-spread $\mathcal{S}'$ properly containing $\mathcal{S}$. Otherwise, $\mathcal{S}$ is called *complete*.

Contributions to the determination of the maximum possible size of partial spreads have been made in [19, 2, 8] and quite recently [29, 32]. For known classifications in the spread case, we refer to references mentioned in [21, Theorem 3.1]. The line spreads in $\mathrm{PG}(5, 2)$ have been classified in [31].

The set of points not covered by the blocks of $\mathcal{S}$ is called the *hole set* and denoted by N. The span $\langle N \rangle$ is called the *hole space* of $\mathcal{S}$. Its dimension will be denoted by n. Often, a good first step for the investigation of partial plane spreads is given by the investigation of the possible dimensions of the hole space and the subsequent classification of the possible hole configurations. For any point set $X \subseteq \begin{bmatrix} V \\ 1 \end{bmatrix}_2$, the number of holes in X is denoted by $h(X)$ and called the *multiplicity* of X. Obviously, $h(X) \leq h(Y)$ for all $X \subseteq Y \subseteq \begin{bmatrix} V \\ 1 \end{bmatrix}_2$, and $h(X) = h(N) = h(V)$ for any subset X containing N. The map $X \mapsto h(X)$ coincides with the extension of the (multi-)set $P \mapsto h(P)$ to the power set of $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ in the sense of [6, Def. 11] and [22, Eq. (18)].

In the following, the number of hyperplanes in $\mathrm{PG}(V)$ containing i blocks is denoted by a_i. The sequence $(a_i)_i$ is called the *spectrum* of $\mathcal{S}$. Often, it is convenient to give spectra in the exponential notation $(1^{a_1} 2^{a_2} \ldots)$, where expressions i^{a_i} with $a_i = 0$ may be skipped.

2.3 Subspace codes

The *subspace distance* on $\mathcal{L}(V)$ is defined as

$$\mathrm{d}(U_1, U_2) = \dim(U_1) + \dim(U_2) - 2\dim(U_1 \cap U_2) = 2\dim(U_1 + U_2) - \dim(U_1) - \dim(U_2)$$

The subspace distance is just the graph-theoretic distance of U_1 and U_2 in the subspace lattice $\mathcal{L}(V)$. Any set $\mathcal{C}$ of subspaces of V is called a *subspace code*. The dimension v of its ambient space V is called the *length* of $\mathcal{C}$, and the elements of $\mathcal{C}$ are called *codewords.* Its *minimum distance* is $\mathrm{d}(\mathcal{C}) = \min(\{\mathrm{d}(B_1, B_2) \mid \{B_1, B_2\} \in \binom{\mathcal{C}}{2}\}$. We denote the parameters of $\mathcal{C}$ by $(v, \#\mathcal{C}, \mathrm{d}(\mathcal{C}))_q$. The maximum size of a $(v, ?, d)_q$ subspace code is denoted by $A_q(v, d)$. See [9] for an overview and [20] for several recent results.

An important class of subspace codes are the *constant dimension (subspace) codes*, where all codewords are subspaces of the same dimension k. In this case, we add the parameter k to the parameters and say that $\mathcal{C}$ is a $(v, \#\mathcal{C}, \mathrm{d}(\mathcal{C}); k)_q$ constant dimension code. The maximum size of a $(v, ?, d; k)_q$ constant dimension code is denoted by $A_q(v, d; k)$. The subspace distance of two k-subspaces U_1, U_2 can be stated as $\mathrm{d}(U_1, U_2) = 2(k - \dim(U_1 \cap U_2)) = 2(\dim(U_1 + U_2) - k)$. In particular, the minimum distance of a constant dimension code is always even.

Setting $t = k - \frac{d}{2} + 1$, we get the alternative characterization of a $(v, ?, d; k)_q$ constant dimension code $\mathcal{C}$ as a set of k-subspaces of V such that each t-subspace is contained in at most one codeword of $\mathcal{C}$. This gives a connection to the notion of a t-$(v, k, \lambda)_q$ subspace design (q-analog of combinatorial designs), which is defined as a subset $\mathcal{D}$ of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ such that each t-subspace is contained in exactly λ elements of $\mathcal{D}$. For example, in this way a $(k-1)$-spread in $\mathrm{PG}(v-1, q)$ is the same as a 1-$(v, k, 1)_q$ design. For our analysis of the hole set of partial spreads, we will start with the spectrum and look at the set of blocks contained in a fixed hyperplane. For (subspace) designs, such a set of blocks is called the *residual design* [26].

As subspace code parameters are preserved under dualization of the subspace lattice of V, we may add dualization to the notion of automorphisms of subspace codes, which yields a group of type $\mathrm{P\Gamma L}(v, q) \rtimes \mathbb{Z}/2\mathbb{Z}$. A subspace code which is isomorphic to its dual will be called *self-dual*.

For the most recent numeric lower and bounds on $A_q(v, d)$ and $A_q(v, d; k)$, we refer to the online tables of subspace codes at `http://subspacecodes.uni-bayreuth.de`. See [17] for a brief manual and description of the implemented methods.

2.4 MRD codes

Let m, n be positive integers. The *rank distance* of $m \times n$ matrices A and B over $\mathbb{F}_q$ is defined as $\mathrm{d_{rk}}(A, B) = \mathrm{rk}(A - B)$. The rank distance provides a metric on $\mathbb{F}_q^{m \times n}$. Any subset $\mathcal{C}$ of the metric space $(\mathbb{F}_q^{m \times n}, \mathrm{d_{rk}})$ is called *rank metric code*. Its *minimum distance* is $\mathrm{d_{rk}}(\mathcal{C}) = \min(\{\mathrm{d_{rk}}(A, B) \mid \{A, B\} \in \binom{\mathcal{C}}{2}\})$. If $\mathcal{C}$ is a subspace of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^{m \times n}$, $\mathcal{C}$ is called *linear*.

If $m \leq n$ (otherwise transpose), $\#\mathcal{C} \leq q^{(m-d+1)n}$ by [5, Th. 5.4]. Codes achieving this bound are called *maximum rank distance (MRD) codes*. In fact, MRD codes do always exist. A suitable construction has independently been found in [5, 13, 34]. Today, these codes are known as the *Gabidulin* codes. In the square case $m = n$, after the choice of a $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$ the Gabidulin code is given by the matrices representing the $\mathbb{F}_q$-linear maps given by the q-polynomials $a_0 x^{q^0} + a_1 x^{q^1} + \ldots + a_{n-d} x^{q^{n-d}} \in \mathbb{F}_{q^n}[x]$. Recently, some progress on the study of MRD codes has been made: The algebraic structure of MRD codes has been analyzed in [4]. New examples of MRD codes have been constructed in [28, 23, 36, 30].

The automorphisms of the metric space $(\mathbb{F}_q^{m\times n}, \mathrm{d}_{\mathrm{rk}})$ are given by the mappings $A \mapsto P\sigma(A)Q + R$ with $P \in \mathrm{GL}(m,q)$, $Q \in \mathrm{GL}(n,q)$, $R \in \mathbb{F}_q^{m\times n}$ and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, and in the square case $m = n$ additionally $A \mapsto P\sigma(A^\top)Q + R$, see [24], [37] and [38, Th.3.4]. The automorphisms of the first type will be called *inner* and denoted by $\mathrm{Inn}(m,n,q)$. The action of these groups provides a notion of (inner) automorphisms and equivalence of rank metric codes. In the non-square case $m \neq n$, any automorphism is inner.

An (inner) isomorphism class X of rank metric codes will be called *linear* if it contains a linear representative. Otherwise, X is called *non-linear*. In a linear isomorphism class, the linear representatives are exactly those containing the zero matrix. Hence, we can check X for linearity by picking some representative $\mathcal{C}$ and some $B \in \mathcal{C}$ and then testing the translated representative $\{A - B \mid A \in \mathcal{C}\}$ of X (which contains the zero matrix) for linearity.

The *lifting* map $\Lambda : \mathbb{F}_q^{m\times n} \to \mathcal{L}(\mathbb{F}_q^{m+n})$ maps an $(m \times n)$-matrix A to the row space $\langle(I_m \mid A)\rangle$, where I_m denotes the $m \times m$ identity matrix. In fact, the lifting map is an isometry $(\mathbb{F}_q^{m\times n}, 2\,\mathrm{d}_{\mathrm{rk}}) \to (\mathcal{L}(\mathbb{F}_q^{m+n}), \mathrm{d})$. Thus, for any $m \times n$ rank metric code $\mathcal{C}$ of size M and minimum distance d, the *lifted code* $\Lambda(\mathcal{C})$ is a $(m + n, M, 2d; m)_q$ constant dimension code. Of particular interest are the lifted MRD codes, which are constant dimension codes of fairly large, though not maximum size.

Fact 2.1 *Let $\mathcal{C}$ be an $m \times n$ MRD code of minimum distance d. Then $\Lambda(\mathcal{C})$ is an $(m+n, q^{(m-d+1)n}, 2d; m)_q$ constant dimension code. Denoting the span of the unit vectors $\mathbf{e}_{m+1}, \ldots, \mathbf{e}_{m+n}$ in $\mathbb{F}_q^{m+n}$ by S, we have $\dim(S) = n$ and each codeword of $\Lambda(\mathcal{C})$ has trivial intersection with S. Moreover, setting $t = m - d + 1$, each t-subspace of V having trivial intersection with S is contained in a single codeword of $\mathcal{C}$.*

2.5 Vector space partitions

A *vector space partition* of V is a set $\mathcal{P}$ of subspaces of V partitioning the point set of $\mathrm{PG}(V)$. The *type* of $\mathcal{P}$ is $(1^{n_1}2^{n_2}\ldots)$, where n_i denotes the number of elements of $\mathcal{P}$ of dimension i. An important problem is the characterization of the realizable types of vector space partitions, see e.g. [15, 7, 16, 35].

A partial $(k - 1)$-spread is the same as a vector space partition of type $(1^{n_1}k^{n_k})$. In Lemma 5.2, we will show that $m \times n$ MRD codes with $m \leq n$ of minimum rank distance m over $\mathbb{F}_q$ are essentially the same as vector space partitions of $\mathbb{F}_q^{m+n}$ of the type $(m^{q^n}n^1)$.

2.6 Partial plane spreads in $\mathrm{PG}(6, 2)$

From now on, we investigate partial plane spreads $\mathcal{S}$ in $\mathrm{PG}(6, 2)$, so we specialize to $v = 7$ and $k = 3$. It is known that the maximum partial plane spreads are of size $A_2(7, 6; 3) = A_2(7, 6) = 17$ [19, 2, 20].

In the following, we prepare some arguments needed later for the classification of the hole structure in the cases $\#\mathcal{S} \in \{16, 17\}$. Of course, the reasoning can easily be translated to general partial spreads.

Lemma 2.2 *Let H be a hyperplane in* $\mathrm{PG}(V)$ *containing i blocks of V. Then* $i \leq \left\lfloor \frac{63-3\#\mathcal{S}}{4} \right\rfloor$ *and* $h(H) = 63 - 3\#\mathcal{S} - 4i$.

Proof. The i blocks cover $\begin{bmatrix}3\\1\end{bmatrix}_2 \cdot i = 7i$ points of H. The intersection of any of the remaining $17 - i$ blocks with H is a line, so their intersections cover $\begin{bmatrix}2\\1\end{bmatrix}_2 \cdot (\#\mathcal{S} - i) = 3(\#\mathcal{S} - i)$ further points of H. The remaining points of H must be holes, so

$$h(H) = \begin{bmatrix}6\\1\end{bmatrix}_2 - 7i - 3(\#\mathcal{S} - i) = 63 - 3\#\mathcal{S} - 4i.$$

The upper bound on i follows since $h(H)$ cannot be negative. □

We have the following system of 3 linear *standard equations* for the spectrum:

Lemma 2.3

$$\sum_i a_i = 127$$
$$\sum_i i a_i = 15\#\mathcal{S}$$
$$\sum_i \binom{i}{2} a_i = \binom{\#\mathcal{S}}{2}$$

Proof. The first equation is simply the observation that each of the $\begin{bmatrix}7\\6\end{bmatrix}_2 = 127$ hyperplane is counted exactly once by the a_i. The second equation arises from double counting the pairs $(H, K) \in \begin{bmatrix}V\\6\end{bmatrix}_2 \times \mathcal{S}$ with $K \leq H$ and the fact that each block is contained in exactly $\begin{bmatrix}7-3\\3-3\end{bmatrix}_2 = 15$ hyperplanes. The third equation arises from double counting the pairs $(H, \{K_1, K_2\}) \in \begin{bmatrix}V\\6\end{bmatrix}_2 \times \binom{\mathcal{S}}{2}$ with $K_1 \leq H$ and $K_2 \leq K$ and the fact that any pair of distinct blocks of $\mathcal{S}$ is disjoint and therefore contained in a unique hyperplane in $\mathrm{PG}(V)$. □

Furthermore, we will make use of the *hole spectrum* $(b_i)_i$ of where b_i is the number of hyperplanes in $\mathrm{PG}(\langle N \rangle)$ containing i holes. The following lemma shows that the hole spectrum is determined by the spectrum and $\dim\langle N \rangle$.

Lemma 2.4 *For all $j \geq 0$,*

$$b_j = \begin{cases} \frac{1}{2^{7-\dim\langle N \rangle}} \cdot a_{(63-3\#\mathcal{S}-j)/4} & \text{if } j < h(\langle N \rangle) \text{ and } j \equiv \#\mathcal{S} - 1 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $b_{h(N)} = 0$, since otherwise there exists a hyperplane T of $\langle N \rangle$ containing all the holes contradicting the definition of the span.

For any hyperplane H in $\mathrm{PG}(V)$ not containing $\langle N \rangle$, the subspace $T = H \cap \langle N \rangle$ is the only hyperplane of $\langle N \rangle$ contained in H. We have that $h(H) = h(T)$.

On the other hand given a hyperplane T in $\mathrm{PG}(\langle N\rangle)$, the number of hyperplanes in $\mathrm{PG}(V)$ containing T, but not $\langle N\rangle$, is

$$\begin{bmatrix}\dim(V)-\dim(T)\\(\dim(V)-1)-\dim(T)\end{bmatrix}_2-\begin{bmatrix}\dim(V)-\dim\langle N\rangle\\(\dim(V)-1)-\dim\langle N\rangle\end{bmatrix}_2$$
$$=\begin{bmatrix}8-\dim\langle N\rangle\\1\end{bmatrix}_2-\begin{bmatrix}7-\dim\langle N\rangle\\1\end{bmatrix}_2=2^{7-\dim\langle N\rangle}.$$

Therefore for all $j<h(N)$,

$$\#\left\{H\in\begin{bmatrix}V\\6\end{bmatrix}_2\mid h(H)=j\right\}=2^{7-\dim\langle N\rangle}\cdot\#\left\{T\in\begin{bmatrix}\langle N\rangle\\\dim\langle N\rangle-1\end{bmatrix}_2\mid h(T)=j\right\}$$

The application of Lemma 2.2 concludes the proof. □

2.7 Scientific software

In the computational parts of our work, The following software packages have been used:

- Computations in vector spaces and matrix groups: magma [3].
- Enumeration of maximum cliques: cliquer [33].
- Enumeration of solutions of exact cover problems: libexact [25], based on the "dancing links" algorithm [18, 27].
- Computation of canonical forms and automorphism groups of sets of subspaces: The algorithm in [11] (based on [10], see also [12]).

3 Maximum partial plane spreads in $\mathrm{PG}(6,2)$

In this section, $\mathcal{S}$ is a maximum partial plane spread in $\mathrm{PG}(V)\cong\mathrm{PG}(6,2)$, i.e. a partial plane spread of size 17. The number of holes is $h(V)=127-17\cdot 7=8$. We are going to prove the following

Theorem 1 *There are* 715 *isomorphism types of maximum partial plane spreads in* $\mathrm{PG}(6,2)$. *For any such maximum partial plane spread* $\mathcal{S}$, *the set of the* 8 *holes is an affine solid* A.

The intersections of the blocks of $\mathcal{S}$ *with the plane* $E=\langle A\rangle\setminus A$ *yield a vector space partition of* E *whose dimension distribution will be called the* type *of* $\mathcal{S}$. *The* 715 *isomorphism types fall into* 150 *of type* (3^1), 180 *of type* $(2^1 1^4)$ *and* 385 *of type* (1^7).

Lemma 3.1 *The spectrum of* $\mathcal{S}$ *is given by* $(1^7 2^{112} 3^8)$.

Proof. By Lemma 2.2, for a hyperplane H containing i blocks of $\mathcal{S}$ we have $i \leq 3$ and $h(H) = 12 - 4i$. Since $h(H)$ cannot exceed the total number $h(V) = 8$ of holes, additionally we get $i \geq 1$. Now Lemma 2.3 yields the linear system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 127 \\ 255 \\ 136 \end{pmatrix}$$

with the unique solution

$$(a_1, a_2, a_3) = (7, 112, 8).$$

□

Lemma 3.2 *The* 8 *holes form an affine solid in* V.

Proof. By $h(H) = 12 - 4i$ and Lemma 3.1, there are $a_1 = 7$ hyperplanes of V containing all 8 holes of $\mathcal{S}$. Since this number equals the number of hyperplanes containing N, we get

$$\begin{bmatrix} 7 - \dim\langle N \rangle \\ 6 - \dim\langle N \rangle \end{bmatrix}_2 = 7$$

with the unique solution $\dim\langle N \rangle = 4$. By Lemma 2.4, the spectrum given in Lemma 3.1 translates to the hole spectrum $(0^1 4^{14})$. Thus, there is a single plane E in $\langle N \rangle$ without any holes. So the affine solid $\langle N \rangle \setminus E$ consists of the 8 holes. □

To reduce the search space to a feasible size, good substructures are needed as starting configurations. For the classification of $(6, 77, 4; 3)_2$ constant dimension codes in [21], "17-configurations" haven proven to provide suitable starting configurations. Modifying this approach for the present situation, we call a set $\mathcal{T}$ of 5 pairwise disjoint planes in $\mathrm{PG}(V)$ a 5-*configuration* if there are two hyperplanes $H_1 \neq H_2$ both containing 3 elements of $\mathcal{T}$. Since $\dim(H_1 \cap H_2) = 5$, in this situation $H_1 \cap H_2$ contains exactly one element of $\mathcal{T}$.

Lemma 3.3 *The partial spread $\mathcal{S}$ contains a* 5*-configuration.*

Proof. By the spectrum, there are 8 hyperplanes containing three blocks. From $3 \cdot 8 = 24 > \#\mathcal{S}$, the 8 sets of blocks covered by these hyperplanes cannot be pairwise disjoint.□

The above lemma allows us to classify the maximum partial plane spreads in $\mathrm{PG}(6, 2)$ by first generating all 5-configurations up to isomorphisms and then to enumerate all extensions of a 5-configuration to a maximum partial spread. For the 5-configurations, we fixed a block B and two hyperplanes $H_1 \neq H_2$ passing through B, which is unique up to isomorphisms. Then, we enumerated all extensions to a 5-configuration by adding two blocks in H_1 and two blocks in H_2 up to isomorphism. We ended up with six types of 5-configurations.

Formulating the extension problem as a maximum clique problem, we computed the number of possible extensions as 2449, 2648, 3516, 3544, 3762 and 25840. Filtering out isomorphic copies, we end up with 715 isomorphism types of partial plane spreads in $\mathrm{PG}(6, 2)$. Their type was determined computationally.

4 Partial plane spreads in $\mathrm{PG}(6,2)$ of size 16

Now $\#\mathcal{S} = 16$ We are going to prove the following result.

Theorem 2 *There are* 14445 *isomorphism types of partial plane spreads in* $\mathrm{PG}(6,2)$ *of size* 16*. Among them,* 3988 *are complete and* 10457 *are extendible to size* 17*.*

(a) The hole set N of the extendible partial plane spreads is the disjoint union of a plane E and an affine solid A.

  (i) In 37 *cases,* $\dim(\langle N\rangle) = 4$ *and* $\dim(E \cap \langle A\rangle) = 3$*. In other words, N is a solid.*

  (ii) In 69 *cases,* $\dim(\langle N\rangle) = 5$ *and* $\dim(E \cap \langle A\rangle) = 2$*. In other words, N is the union of three planes E_1, E_2, E_3 passing through a common line L such that $E_1/L, E_2/L, E_3/L$ are in general position in the factor geometry* $\mathrm{PG}(V/L) \cong \mathrm{PG}(4,2)$*.*

  (iii) In 3293 *cases,* $\dim(\langle N\rangle) = 6$ *and* $\dim(E \cap \langle A\rangle) = 1$*.*

  (iv) In 7058 *cases,* $\dim(\langle N\rangle) = 7$ *and* $\dim(E \cap \langle A\rangle) = 0$*.*

(b) The hole set N of the complete partial plane spreads is the union of 7 *lines $L_1, \ldots, L_7$ passing through a common point P, such that $\{L_1/P, \ldots, L_7/P\}$ is a projective basis of the factor geometry* $\mathrm{PG}(V/P) \cong \mathrm{PG}(5,2)$*. In particular,* $\dim(\langle N\rangle) = 7$*.*

Remark 4.1 The 5 possibilities for the structure of the hole set in Theorem 2 are unique up to isomorphism.

For $\#\mathcal{S} = 16$, the number of holes is $\#N = 15$. By Lemma 2.2, any hyperplane H in $\mathrm{PG}(6,2)$ contains $i \in \{0,1,2,3\}$ blocks and $15 - 4i$ holes of $\mathcal{S}$. So a_3 is the number of hyperplanes containing all the holes of $\mathcal{S}$. Since this equals the number of hyperplanes containing $\langle N\rangle$, a_3 is of the form $\begin{bmatrix}7-\dim\langle N\rangle \\ 6-\dim\langle N\rangle\end{bmatrix}_2 = \begin{bmatrix}7-\dim\langle N\rangle \\ 1\end{bmatrix}_2$. Since $\langle N\rangle$ contains the $15 = \begin{bmatrix}4\\1\end{bmatrix}_2$ holes, necessarily $\dim\langle N\rangle \geq 4$ and therefore $a_0 \in \{0,1,3,7\}$.

Lemma 2.3 yields the following linear system of equations for the spectrum of $\mathcal{S}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 127 \\ 240 \\ 120 \end{pmatrix}$$

Parameterizing by a_0, we get the solution

$$\begin{aligned} a_1 &= 21 - 3a_0 \\ a_2 &= 99 + 3a_0 \\ a_3 &= 7 - a_0 \end{aligned}$$

Plugging in the four possible values $a_0 \in \{0, 1, 3, 7\}$ and applying Lemma 2.4 leads to the following four possibilities.

$\dim\langle N\rangle$	spectrum	hole spectrum
7	$(1^{21}2^{99}3^{7})$	$(3^{7}7^{99}11^{21})$
6	$(0^{1}1^{18}2^{102}3^{6})$	$(3^{3}7^{51}11^{9})$
5	$(0^{3}1^{12}2^{108}3^{4})$	$(3^{1}7^{27}11^{3})$
4	$(0^{7}2^{120})$	(7^{15})

4.1 Hole configuration

The first step for the proof of Theorem 2 is the classification of the hole configuration from the above spectra. The classification will be done entirely by theory, without the need to use a computer. For $n := \dim\langle N\rangle = 4$ the statement immediately follows from $\#\langle N\rangle = \begin{bmatrix}4\\1\end{bmatrix}_q = 15 = \#N$. In the following, we will deal with the cases $n \in \{5, 6, 7\}$, which get increasingly involved.

Proof of Theorem 2, hole structure for $n = 5$. Let $\dim(\langle N\rangle) = 5$. Then the hole spectrum is $(3^{1}7^{27}11^{3})$. Let S_1, S_2 and S_3 be the three solids containing 11 holes.

We show that the three planes $E_1 = S_1 \cap S_2$, $E_2 = S_2 \cap S_3$ and $E_3 = S_3 \cap S_1$ together with $L = S_1 \cap S_2 \cap S_3$ have the claimed properties.

For any two solids S_i and S_j with $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$, $\dim(S_i \cap S_j) = 3$, so $h(E_i) = h(S_i \cap S_j) \leq \begin{bmatrix}3\\1\end{bmatrix}_q = 7$. On the other hand by the sieve formula,

$$h(E_i) = h(S_i) + h(S_j) - h(S_i \cup S_j) \geq 11 + 11 - 15 = 7.$$

So $h(E_i) = 7$, meaning that the plane E_i consists of holes only.

In the dual geometry of $\mathrm{PG}(N)$, the solids S_i are either collinear or they form a triangle. In the first case, $L = E_1 = E_2 = E_3$ consists of holes only and therefore

$$h(S_1 \cup S_2 \cup S_3) = h(S_1 \setminus L) + h(S_2 \setminus L) + h(S_3 \setminus L) + h(L) = 3 \cdot 4 + 7 > 15,$$

a contradiction. So we are in the second case. Here $\dim(L) = 2$, $E_1 + E_2 + E_3 = N$, and the $3 \cdot (7 - 3) + 3 = 15$ points contained in $E_1 \cup E_2 \cup E_3$ are the 15 holes. □

The following counting method is a direct consequence of the sieve formula and will be used several times.

Lemma 4.2 *Let $W \leq Y \leq V$ with $\dim(Y/W) = 2$. Let X_1, X_2, X_3 be the three intermediate spaces of $W \leq Y$ with $\dim(Y/X_i) = 1$. Then*

$$2h(W) = h(X_1) + h(X_2) + h(X_3) - h(Y).$$

We are going to apply Lemma 4.2 to $Y = \langle N\rangle$. Then $h(\langle N\rangle) = 15$ and $h(X_i) \in \{3, 7, 11\}$.

Lemma 4.3 *For each $W \in \begin{bmatrix}\langle N\rangle\\n-2\end{bmatrix}_2$, $h(W)$ is odd.*

Table 1: Codimension 2 hole distributions in $\langle N\rangle$

$h(X_1)$	$h(X_2)$	$h(X_3)$	$h(W)$
11	11	11	9
11	11	7	7
11	7	7	5
11	7	3	3
7	7	7	3
11	3	3	1
7	7	3	1

Proof. Denoting the three codimension 1 intermediate spaces of $W \leq \langle N\rangle$ by X_1, X_2, X_3, Lemma 4.2 yields

$$2h(W) = h(X_1) + h(X_2) + h(X_3) - h(\langle N\rangle) \equiv -1 - 1 - 1 + 1 \equiv 2 \mod 4 \qquad \square$$

As a more detailed analysis, all the possibilities for $(h(X_1), h(X_2), h(X_3))$ (up to permutations) allowed by Lemma 4.2 are listed in Table 1. In particular, the distributions $(7, 3, 3)$ and $(3, 3, 3)$ are not possible as we get the negative numbers $h(W) = -1$ and $h(W) = -3$, respectively. Moreover, $(11, 11, 3)$ is not possible as the resulting $h(W) = 5$ contradicts $W \leq X_3$. Table 1 also shows that in the cases $h(W) \in \{5, 7, 9\}$ the numerical hole distributions are unique.

Proof of Theorem 2, hole structure for $n = 6$. By the hole spectrum $(3^3 7^{51} 11^9)$, $\langle N\rangle$ contains three 4-flats $F_{100}, F_{010}, F_{001}$ of multiplicity 3. Let $E = F_{100} \cap F_{010} \cap F_{001}$. Table 1 shows that $\dim(E) \neq 4$. So $\dim(E) = 3$. The factor geometry $\langle N\rangle/E$ carries the structure of a Fano plane. So we may label the seven intermediate solids by $S_{\mathbf{x}}$ and the seven intermediate 4-flats by $F_{\mathbf{x}}$ with $\mathbf{x} \in \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, such that $S_{\mathbf{x}} \leq F_{\mathbf{y}}$ if and only if $\mathbf{x} \perp \mathbf{y}$. By Table 1, $h(S_{100}) = h(S_{010}) = h(S_{001}) = 1$, $h(F_{011}) = h(F_{101}) = h(F_{110}) = 11$ and $h(S_{111}) = 9$. From $S_{011} \leq F_{100}$, we get $h(S_{011}) \leq h(F_{100}) = 3$ and therefore $h(S_{011}) \in \{1, 3\}$. The application of Lemma 4.2 to $E \leq F_{100}$ gives $h(S_{011}) = 2h(E) + 1$, so $h(E) \in \{0, 1\}$. Doing the same for S_{101} and S_{110}, we arrive at

$$h(S_{011}) = h(S_{101}) = h(S_{110}) = 2h(E) + 1.$$

If $h(E) = 0$, the three solids in F_{111} containing E are of multiplicity 1 and therefore by Table 1, $h(F_{111}) = 3$. This contradicts the hole spectrum, as F_{111} would be a fourth 4-flat of multiplicity 3. So $h(E) = 1$, $h(S_{011}) = h(S_{101}) = h(S_{110}) = 3$ and $h(F_{111}) = 7$.

Let $A = S_{111} \setminus E$. From $h(A) = h(S_{111}) - h(E) = 8$, we see that all the points in A are holes. We denote the single hole in $S_{111} \setminus A$ by P.

We look at the chain $A \leq S_{111} \leq F_{100}$. Out of the 11 holes in F_{100}, 8 are contained in A, the single hole P is contained in $S_{111} \setminus A$ and 2 further holes are contained in $F_{100} \setminus S_{111}$. The three holes in $F_{100} \setminus A$ must be collinear: Otherwise there is a solid S of F_{100} containing 2 of these 3 holes, and as $S \cap A$ is an affine plane, $h(S) = 4 + 2 = 6$ which is not possible by Lemma 4.3. Repeating the argument for F_{010} and F_{001}, we get that $N \setminus A = L_1 \cup L_2 \cup L_3$, where the L_i are lines passing through the common point P. From

$$\begin{aligned} 6 = \dim\langle N\rangle &= \dim(\langle A\rangle + \langle N \setminus A\rangle) \\ &= \dim(S_{111}) + \dim\langle N \setminus A\rangle - \dim(S_{111} \cap \langle N \setminus A\rangle) = 3 + \dim\langle N \setminus A\rangle \end{aligned}$$

we get $\dim\langle N \setminus A\rangle = 3$, so $N \setminus A$ is a plane and all its 7 points are holes. □

Now we partially extend the analysis of Table 1 to $\dim(\langle N\rangle/W) = 3$, characterizing the hole distribution to the intermediate lattice of $W \leq \langle N\rangle$ for "heavy" subspaces W. While this information is only needed for the last case $n = 7$, the proof works for any value of n.

Lemma 4.4 *Let $W \leq \langle N\rangle$ of codimension 3 and $h(W) \geq 6$. Denoting the set of the seven intermediate spaces of codimension 1 by $\mathcal{X}$ and of the seven intermediate spaces of codimension 2 by $\mathcal{Y}$, one of the following two cases arises:*

(i) $h(W) = 8$, $h(X) = 9$ *for all* $X \in \mathcal{X}$ *and* $h(Y) = 11$ *for all* $Y \in \mathcal{Y}$.

(ii) $h(W) = 7$. *There is a single $Y \in \mathcal{Y}$ of multiplicity 7, and the six remaining subspaces in $\mathcal{Y}$ are of multiplicity 11. The three $X \in \mathcal{X}$ contained in Y are of multiplicity 7, the other 4 subspaces in $\mathcal{X}$ are of multiplicity 9.*

In particular, $h(W) = 6$ is not possible.

Proof. For all $X \in \mathcal{X}$, $W \leq X$, so $h(X) \geq h(W) \geq 6$. So only the first two lines in Table 1 are possible, and in particular $h(X) \in \{7, 9\}$ for all $X \in \mathcal{X}$ and $h(Y) \in \{7, 11\}$ for all $Y \in \mathcal{Y}$.

The intermediate lattice of $W \leq \langle N\rangle$ carries the structure of a Fano plane. If there are two distinct $Y_1, Y_2 \in \mathcal{Y}$ of multiplicity 7, Table 1 shows that $Y_1 \cap Y_2 \in \mathcal{X}$ is of multiplicity at most 5, which is a contradiction. So the number of $Y \in \mathcal{Y}$ of multiplicity 7 is either 0 or 1. After several applications of Lemma 4.2 and Table 1, these two possibilities are completed to the stated cases. □

Lemma 4.5 *Let $\mathcal{L}$ be a set of lines in some projective geometry such that no pair of lines in $\mathcal{L}$ is skew. Then at least one of the following statements is true:*

(i) All the lines in $\mathcal{L}$ pass through a common point.

(ii) The lines in $\mathcal{L}$ are contained in a common plane.

Proof. Assume that there is no common point of the lines in $\mathcal{L}$. Then there exist three lines $L_1, L_2, L_3 \in \mathcal{L}$ forming a triangle. Let E be the plane spanned by L_1, L_2 and L_3. Let $L' \in \mathcal{L} \setminus \{L_1, L_2, L_3\}$. Then $P_i := L' \cap L_i \in E$ for all $i \in \{1, 2, 3\}$, and as the three lines L_1, L_2 and L_3 do not pass through a common point, $\#\{P_1, P_2, P_3\} \geq 2$. This implies $L' \leq E$. □

The main step to the classification is the following lemma.

Lemma 4.6 *For $n = 7$, the hole set N contains exactly 7 lines. No pair of these lines is skew.*

Proof. Let ℓ be the number of lines contained in N. We count the set X of pairs $(H, \{P_1, P_2, P_3\}) \in \begin{bmatrix} V \\ 6 \end{bmatrix}_2 \times \binom{N}{3}$ with $\{P_1, P_2, P_3\} \subseteq H$ in two ways. By the hole spectrum $(3^7 7^{99} 11^{21})$,

$$\#X = 7 \cdot \binom{3}{3} + 99 \cdot \binom{7}{3} + 21 \cdot \binom{11}{3} = 6937.$$

On the other hand, each of the ℓ collinear triples of holes generates a 2-subspace, and each of the $\binom{15}{3} - \ell$ non-collinear triples of holes generates a 3-subspace, showing that

$$\#X = \ell \cdot \begin{bmatrix} 7-2 \\ 6-2 \end{bmatrix}_2 + \left(\binom{15}{3} - \ell \right) \begin{bmatrix} 7-3 \\ 6-3 \end{bmatrix}_2 = 16\ell + 6825.$$

Thus $16\ell + 6825 = 6937$ and hence N contains exactly $\ell = 7$ lines. Let $\mathcal{L}$ be the set of these lines.

Assume that $L_1, L_2 \in \mathcal{L}$ are skew. Then $S := \langle L_1, L_2 \rangle$ is a solid. By Lemma 4.4, there are two possible cases.

Case 1 $h(S) = 7$ and S is contained in a hyperplane H of multiplicity 7. Let P be the hole of S which is not covered by the lines L_1 and L_2. Let E be a plane of S passing through L_1 and not containing P. As E intersects L_2 in a point, $h(E) = 3 + 1 = 4$. Since there is no hole in $H \setminus S$, any 4-flat $F \leq H$ with $F \cap S = E$ is of multiplicity $h(F) = h(E) = 4$. This is a contradiction to Lemma 4.3.

Case 2 $h(S) = 8$. So apart from the points on L_1 and L_2, S contains two further holes P_1 and P_2. Each of the points P_i ($i \in \{1, 2\}$) is contained in a single line of $N \cap S$, which is given by the line passing through P_i and the intersection point of L_2 and the plane spanned by P_i and L_1. So in total, S contains 4 lines of $\mathcal{L}$.

By $\#\mathcal{L} = 7$ there exists a line $L_3 \in \mathcal{L}$ not contained in S, implying that L_3 contains at least 2 holes not contained in S. Let $H = \langle L_1, L_2, L_3 \rangle$. Then $h(H) \geq h(S) + 2 = 10$. By Table 1, any 4-flat contains at most 9 holes, so $\dim(H) = 6$, $h(H) = 11$ and by the dimension formula

$$\dim(S \cap L_3) = \dim(S) + \dim(L_3) - \dim(S + L_3) = 4 + 2 - \dim(H) = 0.$$

Therefore, any two of the three lines L_1, L_2 and L_3 are skew.

By the already excluded Case 1, the solids $S' = \langle L_1, L_3 \rangle$ and $S'' = \langle L_2, L_3 \rangle$ are of multiplicity $h(S') = h(S'') = 8$. So each of the solids S, S' and S'' contains two extra holes which are not contained in L_1, L_2 or L_3. As the 11 holes in H are already given by P_1, P_2 and the points on L_1, L_2 and L_3, these two extra holes must be P_1, P_2 for all three solids S, S' and S''. Now $S \cap S'$ contains at least the five holes given by P_1, P_2 and the holes on L_1. Therefore, $\dim(S \cap S') \geq 3$. However, the dimension formula yields

$$\dim(S \cap S') = \dim(S) + \dim(S') - \dim(S + S') = 4 + 4 - \dim(H) = 2.$$

Contradiction. □

Proof of Theorem 2, hole structure for $n = 7$. By Lemma 4.6 and Lemma 4.5, we are in one of the following cases:

(i) The lines in $\mathcal{L}$ pass through a common point P. Thus, $\mathcal{L}$ covers all the $1+7\cdot 2 = 15$ holes, showing that $N = \bigcup \mathcal{L}$. In particular, $\dim\langle \bigcup \mathcal{L}\rangle = n = 7$. So for any set $\mathcal{L}'$ of 6 lines in $\mathcal{L}$, $\dim\langle \bigcup \mathcal{L}'\rangle \in \{6, 7\}$. Indeed, $\dim\langle \bigcup \mathcal{L}'\rangle = 7$, since otherwise $\langle \bigcup \mathcal{L}'\rangle$ is a hyperplane containing at least 13 holes which contradicts the hole spectrum. So $\{L/P \mid L \in \mathcal{L}\}$ is a projective basis of the factor geometry $\mathrm{PG}(V/P)$.

(ii) There is a plane $E \in \begin{bmatrix} V \\ 3 \end{bmatrix}_2$ such that $\mathcal{L} = \begin{bmatrix} E \\ 2 \end{bmatrix}_2$. By $E \subseteq N$, $\mathcal{S} \cup \{E\}$ is a partial plane spread of maximum size 17. By Lemma 3.2, its hole set $N \setminus E$ is an affine solid, showing that N is the disjoint union of a plane and an affine solid. □

4.2 Extendible partial plane spreads of size 16

For the classification of the extendible partial plane spreads of size 16, we make use of the classification of the maximum partial spreads of Section 3.

A partial plane spread $\mathcal{S}$ of size 16 is extendible to size 17 if and only if its hole set contains a plane E. In this case, $N = E \cup A$ where A is an affine solid, $\hat{\mathcal{S}} = \mathcal{S} \cup \{E\}$ is a maximum partial plane spread and its hole set is A. By the dimension formula, $\dim(E \cap \langle A\rangle) = 7 - n$, showing that $7 - n$ must appear in the type of $\hat{\mathcal{S}}$.

The remaining possibilities are:

(i) If $n = 4$, the hole set N contains 15 planes. Extending $\mathcal{S}$ by any of these planes E leads to a maximum partial plane spread $\hat{\mathcal{S}}$ of type (3^1), and E is the unique block contained in the hole space of $\hat{\mathcal{S}}$.

(ii) If $n = 5$, the hole set N contains 3 planes. Extending $\mathcal{S}$ by any of these planes E leads to a maximum partial plane spread $\hat{\mathcal{S}}$ of type $(2^1 1^4)$, and E is the unique block intersecting the hole space of $\hat{\mathcal{S}}$ in a line.

(iii) For $n = 6$, the hole set N contains a single plane. The resulting maximum partial plane spread $\hat{\mathcal{S}}$ is of type $(2^1 1^4)$ or (1^7).

(iv) For $n = 7$, the hole set N contains a single plane. For the resulting maximum partial plane spread $\hat{\mathcal{S}}$, all three types (3^1), $(2^1 1^4)$ and (1^7) are possible.

Remark 4.7 In the cases $n \in \{4, 5\}$, the block E of a maximum partial spread $\hat{\mathcal{S}}$ is called *moving* as it can be exchanged for any of the 15 or 3 other planes preserving the property of $\hat{\mathcal{S}}$ being a maximum partial spread.

Table 2: Types of extendible partial plane spread in $\mathrm{PG}(6,2)$ of size 16

	$\hat{\mathcal{S}}$ of type (3^1)	$\hat{\mathcal{S}}$ of type $(2^1 1^4)$	$\hat{\mathcal{S}}$ of type (1^7)	Σ
$n=4$	37	0	0	37
$n=5$	0	69	0	69
$n=6$	0	604	2689	3293
$n=7$	1324	1890	3844	7058
Σ	1361	2563	6533	10457

Now for the generation of partial plane spreads of size 16 with $n=4$, we go through the 150 maximum partial plane spreads $\hat{\mathcal{S}}$ of type (3^1) and remove the block in the hole space. It is possible that non-isomorphic maximum partial plane spreads yield isomorphic reductions. More precisely, as the hole set of the reduction contains 15 planes, an isomorphism type might be generated up to 15 times in this way. The actual distribution of numbers of reductions falling together is

$$(1^1 2^9 3^8 4^6 5^6 6^2 7^2 9^3).$$

So in total, there are 37 isomorphism types of partial plane spreads with $n=4$. In Theorem 3 and Section 6.2, we will see that these partial plane spreads correspond to vector space partitions of $\mathrm{PG}(6,2)$ of type $(3^{16}4^1)$, to binary 3×4 MRD codes of minimum rank distance 3 (which are given explicitly in Theorem 3) and to $(7,17,6)_2$ subspace codes of dimension distribution $(3^{16}4^1)$.

Similarly, for the generation of partial plane spreads of size 16 with $n=5$, we go through the 180 maximum partial plane spreads $\hat{\mathcal{S}}$ of type $(2^1 1^4)$ and remove the block intersecting the hole space in dimension 2. Since the hole set of the reduction contains 3 planes, an isomorphism type may be generated up to three times. The 180 starting spreads produced the distribution $(1^{10}2^7 3^{52})$ of reductions falling together.

For $n\in\{6,7\}$, we proceed in a similar manner, starting with the maximum partial plane spreads of types $(2^1 1^4)$ and (1^7), or of all three types, respectively. The resulting numbers of isomorphism types of extendible partial plane spreads of size 16 are summarized in Table 2.

4.3 Complete partial plane spreads of size 16

For the remaining classification of the complete partial plane spreads, we fix the hole set N as determined above and let $\mathcal{S}$ be a partial plane spread of size 16 having hole set N. Furthermore, we fix a hyperplane H containing 3 holes and compute the stabilizer G of $N\cup\{H\}$ in $\mathrm{GL}(7,2)$ as a group of order 46080.

By Lemma 2.2, H contains 3 blocks of $\mathcal{S}$. Under the action of G, we find 3 possibilities to add a single plane in H, 18 possibilities to add two disjoint planes and 275 possibilities to add three pairwise disjoint planes in H. For each of these 275 configurations, we enumerate all possibilities for the extension to a partial plane spread of size 16 attaining the hole set N. Stating this problem as an exact cover problem, the running time per

Table 3: The complete partial plane spread $\mathcal{S}_1$ of size 16

$$\begin{pmatrix} 1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&1 \\ 0&0&1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&0&1&0&1&1&1 \\ 0&0&0&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&1 \\ 0&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&1&1 \\ 0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&0&0&1&0 \\ 0&1&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&1&0&0&0 \\ 0&0&0&0&1&0&1 \\ 0&0&0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&0 \\ 0&1&1&0&0&1&0 \\ 0&0&0&1&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&1&1 \\ 0&1&0&1&0&0&0 \\ 0&0&1&0&1&0&0 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&1&1&0&0 \\ 0&1&0&1&1&0&1 \\ 0&0&1&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&1 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&1 \\ 0&1&0&0&1&1&1 \\ 0&0&0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&1&0 \\ 0&0&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&1&0&0&1 \\ 0&1&0&0&0&1&1 \\ 0&0&1&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 0&0&0&0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1&1 \\ 0&1&0&1&1&1&0 \\ 0&0&1&1&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&0 \\ 0&1&0&1&0&1&0 \\ 0&0&1&1&1&1&0 \end{pmatrix}$$

Table 4: The complete partial plane spread $\mathcal{S}_2$ of size 16

$$\begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&1&0&1&1 \\ 0&0&1&1&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&1 \\ 0&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&1 \\ 0&0&0&1&0&0&1 \\ 0&0&0&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&1&0&1&1 \\ 0&0&1&0&0&0&1 \\ 0&0&0&0&1&1&0 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&1&1&1&1 \\ 0&1&0&1&0&0&0 \\ 0&0&1&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&0 \\ 0&1&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0&0 \\ 0&1&1&0&0&0&0 \\ 0&0&0&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0&0 \\ 0&1&0&0&0&0&1 \\ 0&0&0&0&0&1&1 \end{pmatrix},$$
$$\begin{pmatrix} 0&1&0&0&1&1&1 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&1 \\ 0&1&0&0&1&1&0 \\ 0&0&0&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&1 \\ 0&1&0&1&1&0&1 \\ 0&0&1&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0&1 \\ 0&1&1&0&1&0&1 \\ 0&0&0&1&1&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&1&0&0&1 \\ 0&0&1&1&0&1&1 \\ 0&0&0&0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1&1 \\ 0&1&0&1&0&1&0 \\ 0&0&1&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 0&0&1&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&0&0 \\ 0&1&0&1&1&1&0 \\ 0&0&1&1&0&1&0 \end{pmatrix}$$

starting configuration is only a few seconds. The number of extensions per starting configuration ranges between 88 and 2704. In total, we get 66490 extensions. Filtering out isomorphic copies, we end up with 3988 isomorphism types.

Example 4.8 We give the most symmetric complete partial plane spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ as lists of generator matrices in Tables 3 and 4, both having an automorphism group of order 24 isomorphic to the symmetric group S_4. In both cases, the holes are given by the seven lines passing through a unit vector and the all-one vector. The orbit structure of $\mathcal{S}_1$ is $\{1\}\{2\}\{3,4,5,6,7,8\}\{9,10,11,12,13,14,15,16\}$, where 1 denotes the first matrix in the list, 2 denotes the second matrix in the list and so on. The orbit structure of $\mathcal{S}_2$ is $\{1\}\{2,3,4\}\{5,6,7,8\}\{9,10,11,12,13,14,15,16\}$.

5 MRD codes

In this section, we fix the following notation: Let m, n be positive integers. We define $\Phi : \mathrm{Inn}(m,n,q) \to \mathrm{P\Gamma L}(m+n,q)$, mapping an inner automorphism $\phi : A \mapsto P\sigma(A)Q+R$ to

$$\Phi(\phi) : \langle \mathbf{x} \rangle \mapsto \left\langle \sigma(\mathbf{x}) \begin{pmatrix} P^{-1} & P^{-1}R \\ 0 & Q \end{pmatrix} \right\rangle.$$

It is easily checked that Φ is an injective group homomorphism. The image of Φ consists of all automorphisms of $\mathrm{PG}(m+n-1,q)$ fixing the span S of the last n unit vectors.

For all $\phi \in \operatorname{Inn}(m, n, q)$ and all $A \in \mathbb{F}_q^{m \times n}$, we have

$$\begin{aligned}
(\Phi(\phi) \circ \Lambda)(A) &= \Phi(\phi)(\langle (I_m \mid A) \rangle) \\
&= \left\langle \sigma(I_m \mid A) \begin{pmatrix} P^{-1} & P^{-1}R \\ 0 & Q \end{pmatrix} \right\rangle \\
&= \left\langle (I_m \mid \sigma(A)) \begin{pmatrix} P^{-1} & P^{-1}R \\ 0 & Q \end{pmatrix} \right\rangle \\
&= \langle (P^{-1} \mid P^{-1}R + \sigma(A)Q) \rangle \\
&= \langle (I_m \mid R + P\sigma(A)Q) \rangle \\
&= (\Lambda \circ \phi)(A).
\end{aligned}$$

So $\Lambda \circ \phi = \Phi(\phi) \circ \Lambda$.

Lemma 5.1 *(a) Let $\mathcal{C}$ be an $m \times n$ MRD code over $\mathbb{F}_q$. Then the inner automorphism group of $\mathcal{C}$ is given by $\Phi^{-1}(\operatorname{Aut}(\Lambda(\mathcal{C})))$.*

(b) Let $\mathcal{C}$ and $\mathcal{C}'$ be two $m \times n$ MRD codes over $\mathbb{F}_q$. Then $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic as rank metric codes under an inner isomorophism if and only if $\Lambda(\mathcal{C})$ and $\Lambda(\mathcal{C}')$ are isomorphic as subspace codes.

Proof. By Fact 2.1, for any lifted MRD code $\hat{\mathcal{C}}$, the span of the last n unit vectors is the unique subspace S of $\mathbb{F}_q^{m+n}$ of dimension n such that $B \cap S$ is trivial for all $B \in \hat{\mathcal{C}}$. Therefore, any $\psi \in \mathrm{P\Gamma L}(m+n, q)$ mapping a lifted MRD code to another one (or the same) has the form $\Phi(\phi)$ with $\phi \in \operatorname{Inn}(m, n, q)$. The proof is finished using the above statements about Φ. □

By the above lemma, instead of classifying lifted $m \times n$ MRD codes with $m \leq n$ of minimum rank distance d up to inner automorphisms, we can classify constant dimension codes of length $m + n$, dimension m, minimum rank distance $2d$ and size $q^{(m-d+1)n}$ such that each block is disjoint to S.

For $d = m$ these subspace codes are vector space partitions, so we get:

Lemma 5.2 *Let $m \leq n$. Via the lifting map Λ, the inner isomorphism classes of $m \times n$ MRD codes of minimum rank distance m correspond to the isomorphism classes of vector space partitions of type $(m^{q^n} n^1)$ of $\mathbb{F}_q^{m+n}$. The latter are the same as partial $(m-1)$-spreads of size m^{q^n} in $\mathrm{PG}(m+n-1, q)$ whose hole space is of dimension n. Via the map Φ, the inner automorphism group of any of these MRD codes is isomorphic to the automorphism group of the corresponding vector space partition.*

In the particular case of $(m, n) = (3, 4)$, the relevant vector space partitions are given by the partial plane spreads in $\mathrm{PG}(6, 2)$ of size 16 having a hole space of dimension 4, whose number has been determined in Theorem 2. As we are in the non-square case $m < n$, any rank metric isomorphism is inner. We get:

Theorem 3 *(a) There are 37 isomorphism types of vector space partitions of type $(3^{16} 4^1)$ in $\mathrm{PG}(6, 2)$.*

Table 5: Linear binary 3×4 MRD codes of minimum rank distance 3

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 1&0&0&1\\ 0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 1&1&0&1\\ 1&0&1&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0\\ 1&1&0&0\\ 1&0&0&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1\\ 1&0&0&0\\ 1&1&0&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\ 1&0&0&0\\ 1&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\ 0&1&0&0\\ 1&1&0&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 0&1&0&0\\ 1&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1\\ 1&0&0&1\\ 0&1&0&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\ 0&1&1&1\\ 1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\ 1&0&0&0\\ 1&0&1&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

$$\left\langle \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&1&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\ 1&1&0&0\\ 0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1\\ 0&1&0&0\\ 1&0&1&0 \end{pmatrix} \right\rangle_{\mathbb{F}_2}$$

(b) *There are* 37 *isomorphism types of binary* 3×4 *MRD codes of minimum rank distance* 3.

Given the recent research activity on the isomorphism types of MRD codes, it is worth to give a closer analysis:

Theorem 4 *The* 37 *classes of binary* 3×4 *MRD codes of minimum distance* 3 *fall into* 7 *linear and* 30 *non-linear ones. The orders of the automorphism groups of the* 7 *linear ones are* 2688, 960, 384, 288, 112, 96 *and* 64.[1] *Representatives of the* 7 *linear MRD codes in descending order of the automorphism group are shown in Table 5. The orders of the automorphism groups of the* 30 *nonlinear ones are given by the distribution* $(48^3 42^1 36^1 24^4 20^1 18^1 16^1 12^2 9^1 8^2 6^6 4^2 3^2 2^3)$. *Representatives of the* 30 *nonlinear MRD codes in descending order of the automorphism group are shown in Tables 6, 7 and 8.*

[1]By linearity, these automorphisms groups contain the translation subgroup $\{A \mapsto A + B \mid B \in \mathcal{C}\} \cong (\mathbb{F}_2^4, +)$ of order 16.

Table 6: Nonlinear binary 3×4 MRD codes of minimum rank distance 3, part 1

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&0&0\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\end{pmatrix},\\ &\begin{pmatrix}1&1&0&0\\1&1&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&0\\1&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&1&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\1&1&1&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\0&1&1&1\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&1&1\\1&0&1&1\\0&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&1&0&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\0&1&0&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\end{pmatrix},\\ &\begin{pmatrix}1&1&1&1\\0&1&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&1&0\\0&0&1&1\end{pmatrix},\begin{pmatrix}1&1&0&0\\0&0&1&1\\0&1&1&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\1&0&1&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&1&0\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&1&0&1\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&1&1&1\\1&1&1&0\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&1&1&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&1\\0&1&1&1\end{pmatrix},\\ &\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix},\begin{pmatrix}1&1&1&1\\0&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&0&1&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&1&0\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&1&0&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}0&1&1&0\\0&1&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&1&0&0\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&1&1&1\\1&0&1&0\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&1&0&0\\0&1&0&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\0&0&0&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&0&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&1&0\\1&0&1&1\end{pmatrix},\\ &\begin{pmatrix}0&0&1&0\\1&1&0&1\\0&1&0&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&0&1&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&1&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&0&0\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&1&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&0&1\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&1&1\\1&1&1&0\\0&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&1&0&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&0&1&1\\0&0&1&0\end{pmatrix},\\ &\begin{pmatrix}0&1&0&0\\0&0&1&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&1&1&1\\0&0&0&1\\0&1&0&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&1&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&1&1&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&1&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&0&1\\1&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\0&0&1&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&1&0&1\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\1&0&1&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\end{pmatrix},\\ &\begin{pmatrix}1&1&0&1\\1&0&0&0\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&0&0&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&0&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&1&0&1\\1&1&1&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0\\0&1&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&1&1&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}1&1&1&1\\0&0&1&1\\0&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&1\\0&0&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&1&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&1&1&0\\0&0&1&0\end{pmatrix},\\ &\begin{pmatrix}1&1&0&0\\0&1&0&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\1&1&1&1\\0&1&1&0\end{pmatrix},\begin{pmatrix}1&1&0&1\\0&1&1&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1\\1&0&0&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&1&0&1\\1&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&0&1&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&0&1&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&1&1&0\\0&1&0&1\end{pmatrix},\begin{pmatrix}1&0&0&1\\0&1&1&1\\1&0&1&0\end{pmatrix},\\ &\begin{pmatrix}1&1&1&1\\1&1&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&0&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\1&0&0&0\\1&1&1&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&1&1&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\0&1&0&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&1&0&0\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&0&1\\0&1&1&1\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&1&1\\0&1&1&0\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&1\\0&1&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&1&1\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\1&0&0&0\\0&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&1&0\\0&1&0&0\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&0&0\\1&0&0&0\end{pmatrix},\\ &\begin{pmatrix}1&1&1&1\\0&0&0&1\\0&0&1&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&0\\1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&1&1&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&1\\1&1&1&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}1&1&1&0\\0&1&1&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&1&0&1\\1&0&1&1\end{pmatrix}\Big\}\end{aligned}$$

$$\begin{aligned}\Big\{ &\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&1&1\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&1&0&0\\0&1&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&1\end{pmatrix},\begin{pmatrix}0&0&1&1\\0&0&0&1\\1&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&0&1&0\\1&1&1&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\0&0&1&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&1&0&1\\1&1&0&0\end{pmatrix},\\ &\begin{pmatrix}1&1&1&0\\0&1&1&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&1&1&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&0&1&0\\0&1&1&0\end{pmatrix},\begin{pmatrix}1&1&0&1\\1&0&0&0\\1&0&1&1\end{pmatrix},\begin{pmatrix}1&1&0&0\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&0\\1&1&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&1&1&1\\1&0&0&1\\0&1&1&1\end{pmatrix}\Big\}\end{aligned}$$

Table 7: Nonlinear binary 3×4 MRD codes of minimum rank distance 3, part 2

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&1&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&0&0\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&0&1&0\\0&0&1&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}0&0&1&0\\0&1&0&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&0&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&1&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&1&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&0&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&0&1\\0&1&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&0&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&0&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&1&0\\0&1&0&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&1&0&0\\1&0&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&1\\1&1&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&1&1&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&1&1&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&0&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&0&1&0\\0&1&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&0&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&0&0&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&1&0&1\\0&1&1&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&0&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&0&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&1&1&1\\1&1&1&0\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&0&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&0&0&0\\1&0&0&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}0&1&0&1\\1&1&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&1\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&1&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&1&1&1\\1&0&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&1&1&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&0&1&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&1&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&1&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&1&1\\1&0&0&0\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&0&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&0&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&1&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&0&1&1\\1&1&0&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&0&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&1&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&0&1&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&1&0\\1&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&0&1&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&1&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&1&1&0\\1&1&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&1&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&0&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&1&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&0&1&1\\0&1&0&0\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&0&0&1\\1&1&0&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&0&0&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&1\\1&1&1&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&1&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&1&0\\1&0&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&1&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&0&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&0&0&1\\1&1&0&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}0&1&1&0\\1&0&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&1&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&1&0\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&0&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&1&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&1&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&1&1\\1&1&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\0&1&0&0\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&1&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&1\\1&0&1&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1&0&1&1\\1&0&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&0&1&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&1&0&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&1&1&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&0&1&1\\1&1&1&1\end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&0&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&1&0\\0&1&1&1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&0&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&1&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&0&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&0&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&0&1&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&1&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&0&0&0\\1&1&1&1\end{pmatrix} \right\}$$

Table 8: Nonlinear binary 3×4 MRD codes of minimum rank distance 3, part 3

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&0&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&1&1\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&1&0\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&1&0\\1&0&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&0&1\\1&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&0&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&0&1&1\\1&1&1&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&0&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&1&1\\0&1&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&1&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&1&1\\1&0&0&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&1&0\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&0&1&1\\1&0&1&1\end{pmatrix},$$
$$\begin{pmatrix}0&1&1&0\\1&0&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&1&0&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&0&0\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&1&1&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&0&0&1\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&1&0\\0&1&1&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\0&0&0&1\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&0&1&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\0&1&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&1&1&0\\0&1&1&0\end{pmatrix},$$
$$\begin{pmatrix}1&1&0&1\\0&1&0&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&1&0&0\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&0&0&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&1&1&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&1&0\\1&1&1&0\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&1&1&0\\1&1&1&0\end{pmatrix},$$
$$\begin{pmatrix}1&1&1&0\\0&0&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&1&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&1&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&0&1&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&1&1&0\\1&1&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&0&1\\1&1&1&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&0&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&0&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&1&0&1\\0&1&0&1\end{pmatrix},$$
$$\begin{pmatrix}0&1&1&0\\0&0&0&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\1&0&0&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&1&1&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\1&1&0&0\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&1&0\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&1&1&1\\1&0&0&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&1&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&0&0&1\\0&1&1&1\end{pmatrix},$$
$$\begin{pmatrix}1&1&0&1\\0&0&1&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&0&0&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&1&1&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&1&0&1\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&1&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\1&1&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\0&0&1&1\\1&1&0&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&1&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&1&1&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\0&0&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&0\\1&0&1&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&1&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&0\\1&0&1&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&0&1\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&1&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&0&0\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&0&1\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&1&0\\1&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&1&1\\0&1&1&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&1\\0&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&1\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&1&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\1&0&0&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&1&0\\1&1&0&1\end{pmatrix},$$
$$\begin{pmatrix}0&1&1&0\\1&0&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&1\\0&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&0&0\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\1&1&1&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&1&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&0&0&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&0&1&0\\1&1&1&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\end{pmatrix} \Big\}$$

$$\Big\{ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&1&1&0\\1&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&0&1\\1&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&0&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&1\\1&1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1&1\\0&0&1&0\\0&1&1&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&1&1\\0&1&0&1\\1&1&0&0\end{pmatrix}, \begin{pmatrix}1&1&1&0\\0&1&0&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&1&0\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\1&1&0&0\\1&0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}, \begin{pmatrix}1&0&1&1\\1&1&1&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&1\\1&1&1&0\\1&1&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&1\\1&1&0&1\\0&0&1&1\end{pmatrix} \Big\}$$

Table 9: The unique self-dual $(7,34,5)_2$ subspace code

$$\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0&0\\ 0&0&1&0&0&1&0\\ 0&0&0&1&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&1\\ 0&0&1&0&0&0&1\\ 0&0&0&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0&1\\ 0&1&1&0&0&1&1\\ 0&0&0&1&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&1&0\\ 0&1&0&0&0&1&0\\ 0&0&1&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&1\\ 0&1&0&0&1&1&0\\ 0&0&1&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&1&0&0&1&1&1\\ 0&0&0&1&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&1\\ 0&1&0&1&0&0&1\\ 0&0&1&1&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&1&1&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&1&1\\ 0&1&0&1&1&1&1\\ 0&0&1&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0&0\\ 0&1&1&0&1&0&1\\ 0&0&0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&1&0\\ 0&1&0&0&1&0&1\\ 0&0&1&1&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&0&1\\ 0&1&0&1&0&1&0\\ 0&0&1&0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&1&0&0&1&1\\ 0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&1&1\\ 0&0&0&1&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&0&0\\ 0&1&0&0&0&0&1\\ 0&0&1&1&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&1&0\\ 0&0&0&1&0&1&1\\ 0&0&0&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&1&1&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0&0\\ 0&1&0&0&0&1&1\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1&0\\ 0&1&0&1&0&1&0\\ 0&0&1&0&0&1&0\\ 0&0&0&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&1&0&1\\ 0&0&0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1&0\\ 0&1&0&1&0&0&0\\ 0&0&1&1&0&0&1\\ 0&0&0&0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&1\\ 0&1&0&0&0&0&0\\ 0&0&1&1&0&1&1\\ 0&0&0&0&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&1&1\\ 0&0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&0\\ 0&1&0&1&0&0&1\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&0\\ 0&1&0&0&1&0&1\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&1&1&0\\ 0&0&0&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&0&0&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&1&1&0\\ 0&1&1&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&0&0\\ 0&1&0&0&0&1&0\\ 0&0&1&1&0&1&1\\ 0&0&0&0&1&0&0 \end{pmatrix}$$

6 Subspace codes

Based on the classification of partial spreads, in this section the $(7,17,6)_2$ and $(7,34,5)_2$ subspace codes are classified. This completes the work started in [20], where their numbers have already been stated.

Up to dualization the possible dimension distributions for a $(7,17,6)_2$ subspace code are (3^{17}), $(3^{16}4^1)$, and $(3^{16}5^1)$ [20, Theorem 3.2(ii)]. The only possible dimension distribution for a $(7,34,5)_2$ subspace code is $(3^{17}4^{17})$ [20, Remark 4].

We are going to prove the following theorems:

Theorem 5 *There are* 928 *isomorphism types of* $(7,17,6)_2$ *subspace codes. Of these codes,* 715 *are constant dimension codes of dimension distribution* (3^{17})*,* 37 *have dimension distribution* $(3^{16}4^1)$ *and* 176 *have dimension distribution* $(3^{16}5^1)$*.*

Theorem 6 *Up to the action of* $\mathrm{P\Gamma L}(6,2)$ *and dualization, there are* 20 *isomorphism types of* $(7,34,5)_2$ *subspace codes, all of the dimension distribution* $(3^{17}4^{17})$*. Among the* 20 *types of codes, there is a single self-dual one. A representative of this code is given by the row spaces of the matrices in Table 9.*

6.1 $(7,17,6)_2$ subspace codes, dimension distribution (3^{17})

These are exactly the maximum partial spreads in $\mathrm{PG}(6,2)$, whose number of isomorphism types has been determined as 715 in Theorem 1.

6.2 $(7, 17, 6)_2$ subspace codes, dimension distribution $(3^{16}4^1)$

The 16 blocks of dimension 3 form a partial spread $\mathcal{S}_{16}$ in PG$(6, 2)$. The block of dimension 4 has subspace distance at least 6 to all the blocks in $\mathcal{S}_{16}$, implying that its intersection with each block in $\mathcal{S}_{16}$ is trivial. In other words, the block of dimension 4 is the set N of holes of $\mathcal{S}_{16}$. This means that the codes with the given dimension distribution are exactly the vector space partitions of V of type $3^{16}4^1$.

Thus, we have yet another characterization of the 37 objects counted in Theorem 2(i) and Theorem 3 as the types of $(7, 17, 6)_2$ subspace codes of dimension distribution $(3^{16}4^1)$.

6.3 $(7, 17, 6)_2$ subspace codes, dimension distribution $(3^{16}5^1)$

Lemma 6.1 *The $(7, 17, 6)_2$ subspace codes of dimension distribution $(3^{16}5^1)$ are exactly the sets of the form $\mathcal{S}_{16} \cup \{F\}$, where $\mathcal{S}_{16}$ is a partial plane spread of size 16 and F is a 5-space containing all the holes of $\mathcal{S}_{16}$.*

Proof. Let C be a $(7, 17, 6)_2$ subspace code. Then, by the subspace distance 6, we have $C = \mathcal{S}_{16} \cup \{F\}$ with a partial plane spread $\mathcal{S}_{16}$ and a 5-subspace F such that for any block $B \in \mathcal{S}_{16}$, the intersection $B \cap F$ is at most a point. So there remain at least $31 - 16 = 15$ points of F, which are holes of $\mathcal{S}_{16}$. Since the total number of holes is only 15, all the holes have to be contained in F. □

By the above lemma, we can enumerate all subspace codes of dimension distribution $(3^{16}5^1)$ by extending all partial plane spreads $\mathcal{S}_{16}$ of hole space dimension $n = \dim\langle N \rangle \leq 5$ by a 5-flat $F \supseteq N$. Based on the classification of the partial plane spreads $\mathcal{S}_{16}$ in Theorem 2, there are the following two cases:

(i) $n = 5$.
Here $F = \langle N \rangle$ is uniquely determined, so the 69 types of partial plane spreads $\mathcal{S}_{16}$ with $n = 5$ yield 69 types of subspace codes.

(ii) $n = 4$.
Here, F may be taken as any of the seven 5-spaces in PG(V) passing through $\langle N \rangle$. For each of the 37 partial plane spreads $\mathcal{S}_{16}$ with $n = 4$, we check computationally if there arise equivalences among the 7 produced subspace codes. The resulting pattern is

$$(7^1)^3(6^1 1^1)^1(4^1 3^1)^6(4^1 2^1 1^1)^9(4^1 1^3)^1(3^2 1^1)^{12}(2^3 1^1)^2(2^2 1^3)^3$$

For example, the part $(3^2 1^1)^{12}$ means that among the 37 partial plane spreads there are 12, where out of the 7 possibilities for F, 3 lead to the same isomorphism type of a $(7, 17, 6)_2$ subspace code, 3 others lead to a second isomorphism type and a single possibility for F leads to a third isomorphism type. In other words, for 12 of the 37 partial plane spreads, the automorphism group partitions the set of seven 5-flats through N into 2 orbits of length 3 and 1 orbit of length 1. Together, we arrive at 107 types of subspace codes.

In this way, we get:

Theorem 7 *There are* 176 *isomorphism types of* $(7, 17, 6)_2$ *subspace codes of dimension distribution* $(3^{16}5^1)$*. The* 16 *codewords of dimension* 3 *form a partial plane spread in* $\mathrm{PG}(6, 2)$*. In* 107 *cases the hole space dimension is* 4*, and in* 69 *cases the hole space dimension is* 5*.*

6.4 $(7, 34, 5)_2$ subspace codes

As stated above, the dimension distribution is $(3^{17}4^{17})$. Thus, any $(7, 34, 5)_2$ subspace code is the union of a maximal partial spread and the dual of a maximal partial spread in $\mathrm{PG}(6, 2)$.

For their construction, we computationally checked the 715 types of maximal partial spreads from Theorem 1 for the extendibility to a $(7, 34, 5)_2$ subspace code by a maximal clique search. It turned out that this is possible only in 9 cases, with the number of extensions being 16, 768, 192, 192, 2824, 12, 64, 13, and 6. Filtering out PΓL-isomorphic copies, the number of extensions is 1, 2, 6, 6, 2, 2, 4, 13 and 4, comprising 39 types in total. Adding dualization to the group of isomorphisms, we remain with 20 codes. This implies that there is a single self-dual code.

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